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# The Bertrand Model and the Level of Product Differentiation

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***ABSTRACT :*** *Imperfect competition exists in the current economic climate. It can manifest in relation with product quality, quantity (Cournot type), orprice ( Bertrand type) . This paper intent is to analyze a duopoly market where both firms adopt a Bertrand behavior. Regardless the product's differentiation level, both firms are expected to survive and a stable equilibrium will manifest. In the case of a non differentiation scenario (homogeneous products), the price will match the marginal cost, identical quantities will be sold and aggregate profit will be zero, situation known as Bertrand's Paradox.*
***Keywords****: Bertrand model ,Bertrand paradox,Oligopoly,Product differentiation*

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## I. INTRODUCTION

Oligopolytheory has a long and distinguished history. Dating back two centuries ago, first studies identify the gradual evolutionof this theory with a traditional stage, where monopoly&competitivebehaviors were analyzed, followed by a later one, where games theory was applied for a better understanding of the oligopoly behaviors (John von Neumann and Oskar Morgenstern - (1944)) whilst various oligopoly models were created/improved to mirror real market conditions (see Joe Bain (1956), Paolo SylosLabini (1957) and Franco Modigliani's papers (1958)).

As representatives of the traditional stage,A. Cournot and J.L.F. Bertrand's models stand out (both scientists being later named by Xavier Vives "co-founding fathers of oligopoly theory" (2001)). Cournot presents a duopoly scenario, with firms producing homogeneous products, and competing in quantities, while Bertrand was advocating price competition. Although was initially written as a review of Cournot's theory, Bertrand's approach (1883) has become the most used model in price competition scenarios. It's main assumptions were: the existence of at least two competing firms producing homogeneous products, equal awareness of market demand,price competition scenario, price being simultaneously set up by the firms withconsumers choosing to buy from the one who's offering the lowest price, or equally from all/each of them, in matching price context.

Current oligopoly literature contain various studies based on Bertrand model. Using Dixit's general framework (1979), Singh &Vives (1984) highlight quantity competition (substitute products) and price competition (complementary products) as dominant strategies.Hackner (2000), Zanchettin (2006) and Tremblay (2011) are adopting a different approachwith informational asymmetry (including demand's asymmetry) triggering optimality of Bertrand or mixed Cournot-Bertrand models. Regardless the approach,cost and demand function linearity were the common hypotheses of the majority of the studies (Ahmed et all (2006), Zhang et all (2009), Tremblay (2011)); demand non-linearity wasanalyzed by Ahmed, Alsadany&Puu (2015), whilst Yi & Zeng (2015)were looking at thecost function non-linearity.

The so-called Cournot-Bertrand duality theory, first time mentioned by Sonnenschein (1968), represents another important step in the development of the oligopoly theory.It offers the dual perspective of theCournot/Bertrand competition (substitute products scenario) respectively the Bertrand/Cournot competition (complementary products scenario), having the same strategic properties (Singh &Vives, 1984). Studying one model should becomprehensiveenough, as the other one will follow similar principles.

The next section will investigate the impact of product differentiation on Bertrand static equilibrium model, highlighting some interesting aspects such as firm stability, survival potential, as well as product differentiation impact on Nash equilibrium theory. The principles of the related mathematic model are also presented in the paragraphs below.

## II. THE MODEL

The background used is one with high number of consumers but only two producers of differentiated good. It further analyzes the potential market equilibrium, with consumers aiming to maximize their own satisfaction; this is described as thedifference between own utility function and pricefor purchasing required product quantities, with no budgetary constraints:

$$S = U(q_1, q_2) - \sum_{i=1}^{2} p_i\, q_i \quad (1)$$

Mathematically, the utility function is considered to be quadratic (non-linear), with separable variables and also strictly concave, as per bellow:

$$U(q_1, q_2) = \alpha_1 q_1 + \alpha_2 q_2 - \frac{\beta_1 q_1^2 + 2dq_1q_2 + \beta_2 q_2^2}{2}$$

where$\alpha_i > 0, \beta_i > 0, d\epsilon[0;1],\ \beta_1\beta_2 - d^2 > 0, \alpha_i\beta_j - \alpha_j d > 0\ (\forall)\ \ i = \overline{1,2}$

The above hypothesis automatically involves double derivability,the existence and also the negativity of the second order derivate. The starting point in duopoly demand function calculation is represented by the derivation of the consumer satisfaction function.

$$\boldsymbol{\frac{\partial S}{\partial q_1} = \alpha_1 - \beta_1 q_1 - dq_2 - p_1 \frac{\partial S}{\partial q_2} = \alpha_2 - \beta_2 q_2 - dq_1 - p_2}$$

First order conditions triggersthe linearity of the demand function, whose inverse are :

$$p_1 = \alpha_1 - \beta_1 q_1 - dq_2 p_2 = \alpha_2 - \beta_2 q_2 - dq_1$$

applied to quantity values that allow price positivity. Further:

$$q_1 = \frac{\alpha_1 - p_1 - dq_2}{\beta_1} q_2 = \frac{\alpha_2 - p_2 - dq_1}{\beta_2}$$

Applying substitution methodology, will result:

$$q_1 = \frac{\alpha_1}{\beta_1} - \frac{p_1}{\beta_1} - \frac{d}{\beta_1} * \frac{\alpha_2 - p_2 - dq_1}{\beta_2} \Rightarrow q_1\left(1 - \frac{d^2}{\beta_1\beta_2}\right) = \frac{\alpha_1}{\beta_1} - \frac{p_1}{\beta_1} - \frac{\alpha_2 d}{\beta_1\beta_2} + \frac{dp_2}{\beta_1\beta_2}$$

$\frac{d^2}{\beta_1\beta_2}$fractionis extremely important,reflecting the degree of product differentiation; zero value indicates independent products, whilst unitary value is specific to homogenous products.
Demand functions will become:

$$q_1 = \frac{\alpha_1\beta_2 - \alpha_2 d}{\beta_1\beta_2 - d^2} - \frac{\beta_2}{\beta_1\beta_2 - d^2} * p_1 + \frac{d}{\beta_1\beta_2 - d^2} * p_2$$

$$q_2 = \frac{\alpha_2\beta_1 - \alpha_1 d}{\beta_1\beta_2 - d^2} - \frac{\beta_1}{\beta_1\beta_2 - d^2} * p_2 + \frac{d}{\beta_1\beta_2 - d^2} * p_1$$

under the same positivity restriction. ”d"indicates the nature of the products, positive values for substitute products, negatives values for complements, with zero values representing independent products. Demand function for i product, decreases in relation to its price, but increases/decreases in substitute/complement products scenario.
Using$\alpha_1 = \alpha_2 = a, \beta_1 = \beta_2 = 1$ as assumptions, the utility functionbecomes:

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{q_1^2 + 2dq_1q_2 + q_2^2}{2} (2)$$

being expected to determine a linear demand functions whichinverse is:

$$p_1 = a - q_1 - dq_2 \rightarrow q_1 = \frac{a(1-d)}{1-d^2} - \frac{1}{1-d^2} * p_1 + \frac{d}{1-d^2} * p_2$$
$$p_2 = a - q_2 - dq_1 \rightarrow q_2 = \frac{a(1-d)}{1-d^2} - \frac{1}{1-d^2} * p_2 + \frac{d}{1-d^2} * p_1$$

a system similar to those usedbefore by  Dixit (1979), Singh &Vives (1984), Imperato et all  (2004), Tremblay (2011).

It can be noted the necessity that d≠1 at this stage.
As for the production cost, this is considered identical for both players, expressed by a linear function (C = c * q) and also matching the marginal cost. Based on these assumptions, the profit function become:

$$\pi_i = (p_i - c)q_i\ , (\forall)i = \overline{1,2}$$

Marginal profits as well as Appendix A calculations, leads to Nash equilibrium values:

$$p_1^* = p_2^* = \frac{a(1-d)+c}{2-d} (3) q_1^* = q_2^* = \frac{a-c}{(1+d)(2-d)}\ (4) \qquad \pi_1^* = \pi_2^* = \frac{(a-c)^2(1-d)}{(2-d)^2(1+d)}\ \ (5)$$

At this point, we can formulate the following initial conclusions:
- If d = 0 the model confirms that both players act as monopolists;
- Both firms have same Nash equilibrium behavior (values);
- If d increases up to 1, price and profit decrease,equilibrium becoming more competitive.

To furtheranalyze the Nash equilibrium stability, we need to start with Dixit's necessary and sufficient stability condition(1986):$|\pi_{ii}| > |\pi_{ij}|\, , where \pi_{ii} = \frac{\partial^2 \pi_i}{\partial p_i^2} iar \pi_{ij} = \frac{\partial^2 \pi_i}{\partial p_j^2}\ ,\ i,j = \overline{1,2}$

$$\begin{cases} \dfrac{\partial^2 \pi_1}{\partial p_1^2} > \dfrac{\partial^2 \pi_1}{\partial p_1 p_2} \\ \dfrac{\partial^2 \pi_2}{\partial p_2^2} > \dfrac{\partial^2 \pi_2}{\partial p_2 p_1} \end{cases} \rightarrow \left|\frac{-2}{1-d^2}\right| > \left|\frac{d}{1-d^2}\right| \rightarrow \frac{1}{1-d^2} > \frac{d}{2(1-d^2)} \xRightarrow{d \epsilon (0;1)} 2 > d(A).$$

**Conclusion:**equilibrium is stable$(\forall)\ d \in (0;1)$.According to Mas-Colell (1995), a static model equilibrium is stable when the "adjustment process, in which the firms take turns myopically playing a best response to each other's currentstrategies, converges to the Nash equilibrium from any strategy pair in a neighborhood of the equilibrium".

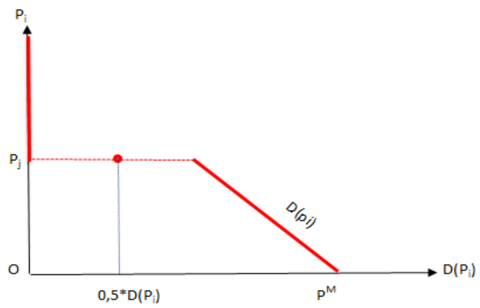
Next paragraphs will analyze the d = 1 - perfectly substitutes products scenario. Therefore we have$\frac{\partial U}{\partial q_i} = a - q_i - q_j = p_i,\ (\forall)i,j = \overline{1,2}$ , then $p_i = p_j$=pand further$q_i + q_j = a - p$. Consumers will choose to buy at the lowest price, however price being identical and no individual preferences is manifested, market demand will be perfectly split between the two producers. Thus$q_i = q_j = \frac{a-p}{2}$ (6), profit becomes$\pi_i = (p-c)q_i = (p-c)\frac{a-p}{2} = \frac{ap - p^2 - ac + cp}{2}$. Having as start point the first order condition,the mathematical calculation leads to:

$$p^* = c (7) \pi^* = 0\ (8)$$

Nash equilibrium istranslated inprofit maximization for the playeri, regardless player j behavior, conclusion mathematically expressed below:

$$\begin{cases} \pi^i(p_i^*, p_j^*) \geq \pi^i(p_i, p_j^*)(\forall)i,j = \overline{1,2} \\ \pi^j(p_i^*, p_j^*) \geq \pi^j(p_i^*, p_j)\ (\forall)\ i,j = \overline{1,2} \end{cases}$$

**Proposition**: $p_1 = p_2 = c$ and$\pi_1^* = \pi_2^*$ =0 defines the only Nash equilibrium.
Proof: as we have already mentioned, demand for product idepends on the price set up by the othercompetitor (Machado, Economia Industrial) and is expressed as follows:




$$D(p_1, p_2) = \begin{cases} D(p_1); p_1 < p_2 \\ \dfrac{D(p_1)}{2}; p_1 = p_2 \\ 0;\ \ p_1 > p_2 \end{cases}$$

**Figure 1**:Firm's i demand function and its dependence of rival's price

In any duopoly scenario, we may have one of the following noted scenarios:

a) $P_1^* > P_2^* > c$ . Thus $D(p_1) = 0 \rightarrow \pi_1 = 0$ , $D(p_2) = D(p_2^*) \rightarrow \pi_2 = (p_2^* - c)D(p_2) > 0$
First player best response would have been $P_1' = P_2^* - \varepsilon$ ,generating positive profit.
b) $P_1^* = P_2^* > c$. In this case $\pi_1^* = (p_1^* - c)\frac{D(p_1)}{2}$ , $\pi_2^* = (p_2^* - c)\frac{D(p_2)}{2}$
First player best response would have been $P_1' = P_2^* - \varepsilon$ which would lead to the seizure of the entire demand, so $D(p_1) = D(p_1')$ therefore $\pi_1' = (p_1' - c)D(p_1') > (p_1^* - c)\frac{D(p_1)}{2} = \pi_1^*$
c) $P_1^* > P_2^* = c$. Then $D(p_1) = 0 \rightarrow \pi_1 = 0,\ D(p_2) = D(p_2^*) \rightarrow \pi_2 = (c - c)D(p_2) = 0$
Second player best response would be $P_2' = P_1^* - \varepsilon$ and $\pi_2' = (p_2' - c)D(p_2') > 0 = \pi_2^*$
d) $P_1^* = P_2^* = c$. Then $D(p_1) = D(p_2) = \frac{D(p_1,p_2)}{2} \rightarrow \pi_1 = \pi_2 = (c - c)\frac{D(p_1,p_2)}{2} = 0$
If $P_1$ □ ➔ $\pi_1 = (p_1^* - \varepsilon - c)D(p_1 - \varepsilon) < 0 = \pi_1^*$ and if $P_1$ □ ➔ $P_1 > P_2$ ➔ $D(p_1) = 0 = \pi_1^*$. Any action path first player would take, would lead to not a higher profit level then the one expected from its current strategy, therefore he is not motivated to modify the price triggering the unique Nash equilibrium point.

**Conclusion:**in case of homogeneous products (perfectly substitutable), equilibriumis stable, the price willequalmarginal cost - at which both players offer half of the existing market output, whilst aggregate profit is zero–scenario known in specialized literature as the Bertrand Paradox.

Optimal response of player i to player jactions, is described by his reaction function:

$$R_i(p_j) = \begin{cases} p_M; p_j > p_M \\ p_j - \varepsilon;\ c < p_j \le p_M \\ c; p_j \le c \end{cases}$$

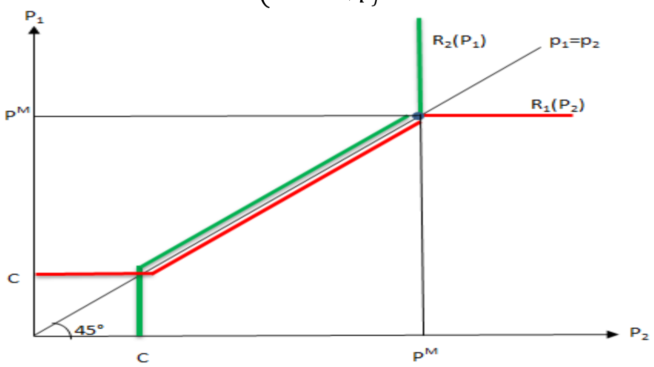



**Figure 2:**Duopolist's reaction functions

We furtheranalyze, via graphical representation, the price/quantity/profit sensitivity to the changes in the level of product differentiation (d parameter values) in a Nash equilibrium scenario. Using the Appendix B as starting point and customizing parameters a and c( a = 80 EUR,  c = 30 EUR) we've gradually increased product homogeneity degree by ratio of 0.05 (from theindependent products scenario (d = 0) to homogeneous products one (d = 1))

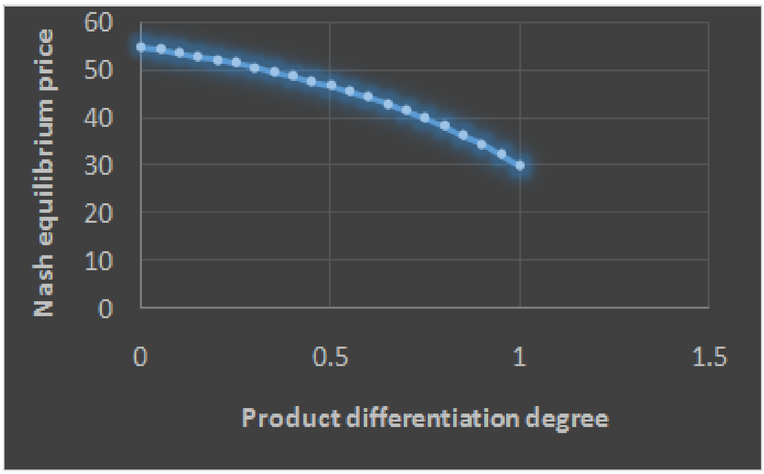



**Figure 3**:Nash equilibrium price evolution

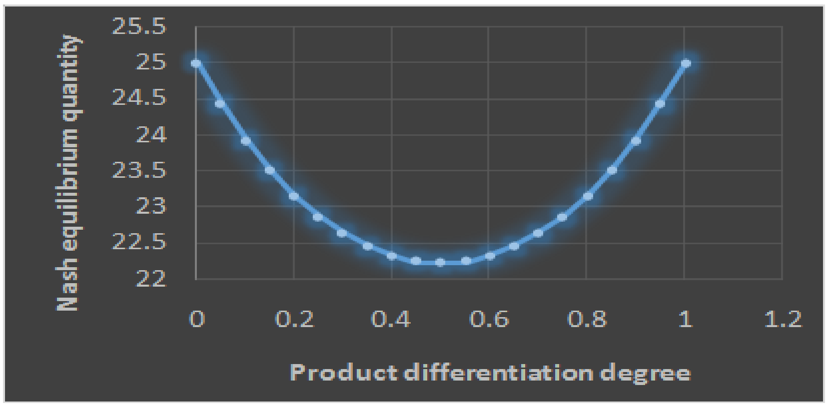



**Figure 4**: Nash equilibrium quantity evolution

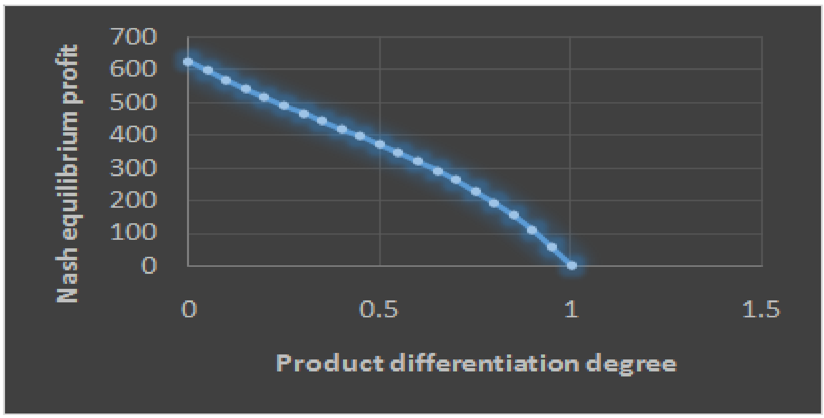



**Figure 5:**Nash equilibrium profit evolution

**Conclusions:**In independent products case (d = 0), the coefficients of a and c are equal, following opposite trendlines as the degree of products differentiation decreases, although their sum remains unitary, as$\frac{1-d}{2-d} + \frac{1}{1-d} = 1$. As a>c, we are witnessingthe gradual price decrease, from a and c average value of 55 EUR, down to marginal cost level of 30 EUR;

As for the quantity triggering the equilibrium scenario,thecoefficients distribution symmetry can be noted in(0;1) interval. Variations are not high, oscillating between maximum value 0.5 (tangible in interval corners) and 0,(4). The explanation is also mathematical (Appendix C), referring to the fact that for$q^{*'} = -\frac{(a-c)(1-2d)}{(1+d)^2(2-d)^2}$the unique critical point(also minimum point) is d=0.5.Then the function shows a decreasing trendline before and and increasing trendline after; the quantity equilibrium level is gradually decreasing from its initial 25 items equilibrium value, bouncing back in homogenous products scenario.

Profit for equilibrium scenario has a downward trend, starting from 0.25 $(a-c)^2$ down to zero value for homogeneous products (so-called Bertrand paradox). Math principles, has one more time to be noted as$\pi^{*'} = -\frac{2(a-c)^2(d^2-d+1)}{(1+d)^2(2-d)^3}$, strictly negative expression (Appendix D) reflecting a decreasing function.Moreover the graphical analyse showsa decreasing profit trend from 625 EUR down to the breakeven point (zero profit).

## III. GRAPHIC APPROACH

The model can be also explained by using a graphical approach, based on duopolist'sreaction functions. The isoprofit curves are convex to the axes (measuring players prices). Each isoprofit curve shows a constant level of profit that could be obtained by the first player (player A) at different price levels charged by him and his competitor (player B).

First player convex isoprofit curve, reflects the need of adjusting its own price down to a certain level (figure 3) to face his rival's price cut, while also maintaining the same profit level on curve $\Pi_{A2}$. Once this level reached, if player B continues to reduce its price, player A will not be able to retain its profits, even if he decides to keep the price at the same level ($P_{Ae}$). For example, if company B reduces the price to $P_B$, company A will move to a low-level isoprofit curve ($P_{A1}$), as result of price decrease and also production increase beyond the optimal plantutilization level( involving cost increases).

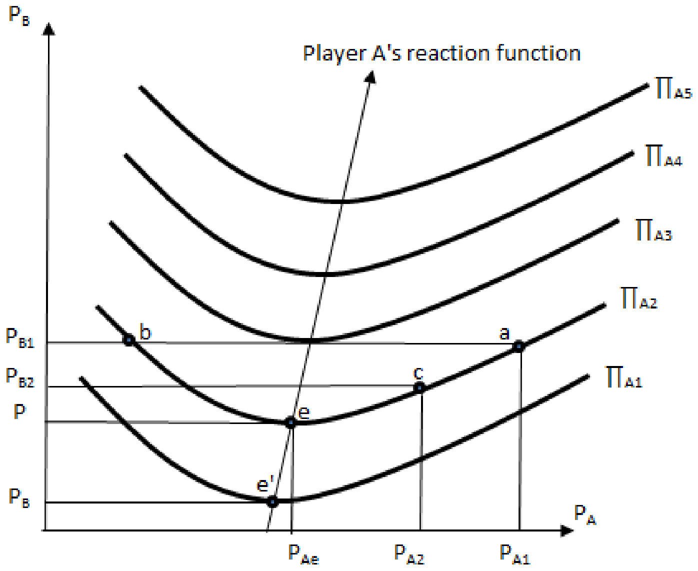



**Figure 6**:A player's reaction function and its isoprofit curves

In summary,for any price charged by player B there will be a uniqueprice firm A can charge, in order to still be able to maximize its profit. From a graphical perspective, moving on a higher profit curve involvesthe minimum point movement to the right, as a result of seizing some of B's customers, due to his decision to increase the charged price, even if player A does the same.

By joining the lowest points of the isoprofit curves, we obtain the reaction function of player A, meaning the geometrical place of his maximum profit levels, at a certain charged price, depending on the price of his rival. Player B's reaction curve can be similarly determined by minimum points of his isoprofit curves jointure (Figure 7).

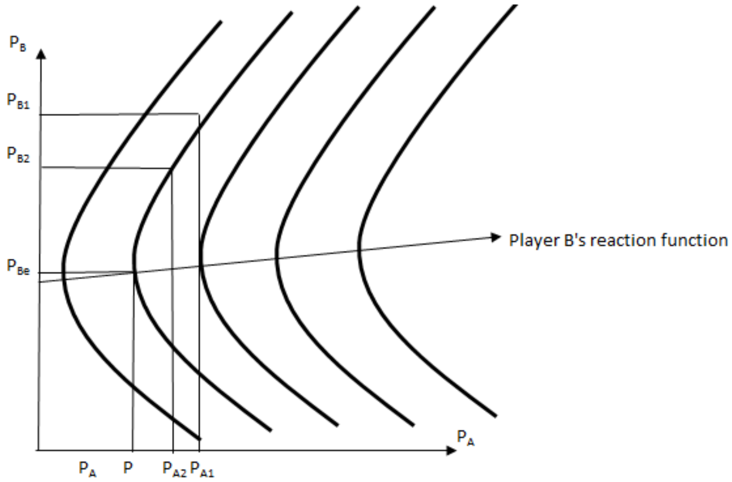



**Figure 7**: B player's reaction function and its isoprofit curves

Based on the above noted points, we can conclude that Bertrand model equilibrium is stable (reached in point e); any deviationwill determine successive movements that will bounce back in the same equilibrium point. For example, if player A sets a lower price than $P_{Ae}$ level ($P_{A1}$), player B will charge $P_{B1}$, as in Bertrand's assumptions it will be a profit maximizer. A's answer will be a higher $P_{A2}$, where B will react again via $P_{B2}$ and so on, until it arrives at point e, representing market's equilibrium. The same equilibrium will be achieved if first player initially charges a higher price than equilibrium level: the answer is $P_{B1'}$ is followed by a fall to $P_{A2}$'and then $P_{B2'}$ response, etc., the competitive price cut bouncing back to the $P_{Ae}$ and $P_{Be}$ equilibrium levels intersection.

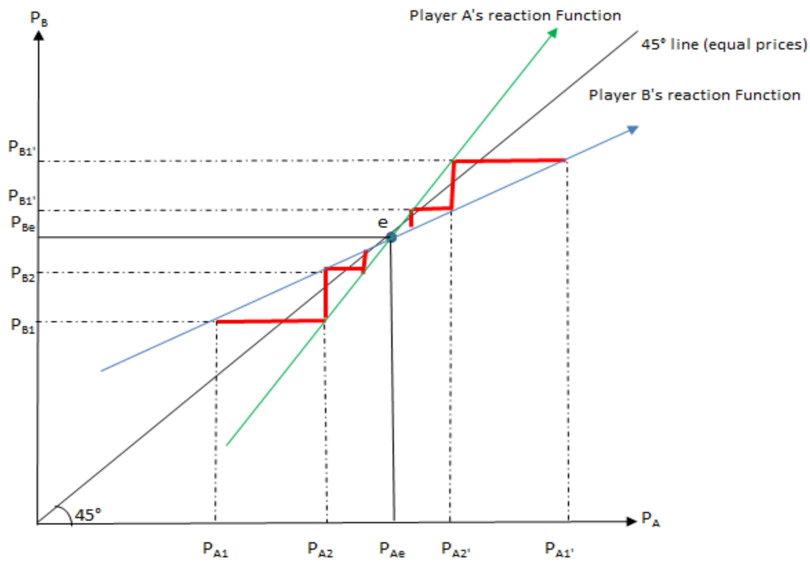



**Figure 8**:Bertrand equilibrium

What is really important to remember is that the Bertrand model does not maximize aggregate profit, as players behave naively, never learning from past experiences, assuming that their rival will not change price level. Industry profits could only be increased if firms will recognize previous errors and stop adopting Bertrand's behavior.

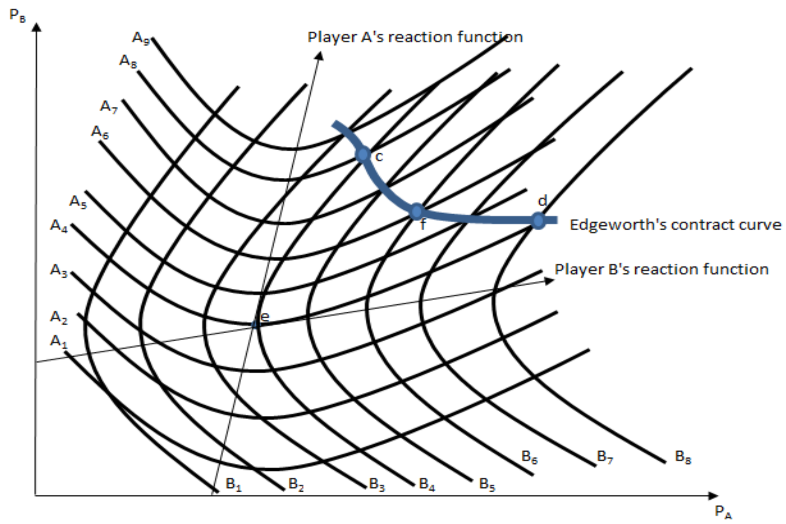



**Figure 9:** Rational player'sequilibrium and Edgeworth contract curve

Figure 9presents player's behavior in the above mentioned scenario. The blue part of the graph represents Edgeworth's contract curve, more specifically the geometric location of the tangent points of both competitors' isoprofit curves.

It can be noted that in point c, player B would register same profit level ($B_4$) as in point e, while player A will move to a higher profit level ($A_8$). In point d, player A would have same profit level ($A_4$) as the Bertrand equilibrium, while B would move to a superior isoprofit curve ($B_8$). At any other point between c and d (such as f), both firms would achieve higher profits ($A_6$ and $B_6$ curves) compared to those obtained with Bertrand's solution ($A_6 > A_4$ and $B_6 > B_4$), therefore profits from industry would be higher.

To conclude our analyse, is perhaps useful to mention Bertrand's model weaknesses, which over time, have become the subject of many criticism from experts (just like the Cournot model):

• The behavior pattern is naive: firms never learn from past experience. Each company aims to maximize their own profits, but aggregate profits are never maximized.

• The equilibrium price will be the competitive price; if we consider some particular caseswith no cost production, (ex. mineral water, fishing bait, etc.)the price should fall to zero; in a no-costless scenario, the price should cover duopolists' costs, as well as a normal profit.

• The model is "closed"-entry barriers exist, their level directly influences the company's ability to increase its profits.

An interesting observation for bothBertrand and Cournot models is that their limit is pure competition. They validate each other, both are consistent, based on different behavioral assumptions. We may say that Bertrand's assumptions are more realistic, existing a higher probability that a supplier will focus on price rather than quantity (excepting inflation case).

• If there is a duopoly situation in a particular market, we can consider the possibility of tacit collusion, or at least a quiet industry, meant to avoid a price war.

• The motivational system of buyers is not limited at choosing the cheaper product. In their decision they will consider some other factors toosuch as product quality, the convenience of using it, purchase simplicity, brand loyalty, etc.

• Serious limitations are naive behavioralrival's pattern, failure to deal with the entry,the inability to incorporate other variables into the model, such as advertising and other selling activities, plant location and product changes.

Product differentiation and sales activities are the two main non-price competition weapons, which represent a main form of competition in the business world; both models do not define the length of the adjustment process.Although it refers todynamicbehavior, the approach is basically static: perfect awareness of market demand is assumed; individual demand curves can be identified making convenient assumptions of the competitors' constant reaction curves

**Appendix A**

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = m - 2np_1 + lp_2 + nc = 0 \\ \frac{\partial \pi_2}{\partial p_2} = m - 2np_2 + lp_1 + nc = 0 \end{cases} \Rightarrow \begin{cases} p_1 = \frac{m+lp_2+nc}{2n} = \frac{a-ad+dp_2+c}{2} \\ p_1 = \frac{2np_2-m-nc}{l} = \frac{2p_2-a-c+ad}{d} \end{cases}$$

where$m = \frac{a(1-d)}{1-d^2}$ , $n = \frac{1}{1-d^2}$, $l = \frac{d}{1-d^2}$. By substitution:

$$\frac{d}{1-d^2}\frac{a-ad+dp_2+c}{2} = -\frac{a-ad}{1-d^2} + \frac{2p_2}{1-d^2} - \frac{c}{1-d^2} \to ad - ad^2 + d^2p_2 + cd =$$

$$= -2a + 2ad + 4p_2 - 2c \to p_2(4-d^2) = -ad(1+d) + 2a + 2c + cd.$$

Therefore$p_2^* = \frac{-ad(1+d)+2a+2c+cd}{4-d^2} = \frac{a(1-d)+c}{2-d}$ and similarly$p_1^* = \frac{a(1-d)+c}{2-d} = p_2^*$

Equilibrium prices are identical. Identifying the appropriate quantities involve:

$$\begin{aligned} q_1^* = m - np_1^* + lp_2^* &= \frac{a(1-d)}{1-d^2} + \frac{d-1}{1-d^2}\frac{-ad(1+d)+2a+2c+cd}{4-d^2} \\ &= \frac{4a-4ad-ad^2-ad^3+ad}{(1-d^2)(4-d^2)} + \frac{ad^2-2a-2c-cd-ad^2-ad^3+2ad+2cd+cd^2}{(1-d^2)(4-d^2)} \\ &= \frac{(a-c)(2-d-d^2)}{(1-d^2)(4-d^2)} = \frac{a-c}{(1+d)(2-d)} = q_2^* \end{aligned}$$

The equilibrium quantities are equals for the two players. At this point, we can calculate the profit obtained in the Nash equilibrium scenario:$\pi_1^* = \pi_2^* = (p^*-c)q^* = \frac{-ad^2-ad+2a+2c+cd-4c+cd^2}{4-d^2} * \frac{(a-c)}{(1+d)(2-d)} = \frac{(a-c)(2-d-d^2)}{4-d^2} * \frac{(a-c)}{(1+d)(2-d)} = \frac{(a-c)^2(1-d)}{(2-d)^2(1+d)}$

**Appendix B**

**Table 1**: Simulation of price, quantity and profit evolution

| d | p | q | ∏ |
|---|---|---|---|
| 0 | 0.5*a+0.5*c | 0.5*(a-c) | 0.25*(a-c)$^2$ |
| 0.05 | 0.487179*a+0.512821c | 0.4884*(a-c) | 0.237939*(a-c)$^2$ |
| 0.1 | 0.473684*a+0.526316*c | 0.478469*(a-c) | 0.226643*(a-c)$^2$ |
| 0.15 | 0.459459*a+0.540541*c | 0.470035*(a-c) | 0.215962*(a-c)$^2$ |
| 0.2 | 0.444444*a+0.555556*c | 0.462963*(a-c) | 0.205761*(a-c)$^2$ |
| 0.25 | 0.428571*a+0.571429*c | 0.457143*(a-c) | 0.195918*(a-c)$^2$ |
| 0.3 | 0.411765*a+0.588235*c | 0.452489*(a-c) | 0.186319*(a-c)$^2$ |
| 0.35 | 0.393939*a+0.606061*c | 0.448934*(a-c) | 0.176853*(a-c)$^2$ |

| | | | |
|---|---|---|---|
| 0.4 | 0.375*a+0.625*c | 0.446429*(a-c) | 0.167411*(a-c)$^2$ |
| 0.45 | 0.354839*a+0.645161*c | 0.444939*(a-c) | 0.157882*(a-c)$^2$ |
| 0.5 | 0.333333*a+0.666667*c | 0.444444*(a-c) | 0.148148*(a-c)$^2$ |
| 0.55 | 0.310345*a+0.689655*c | 0.444939*(a-c) | 0.138084*(a-c)$^2$ |
| 0.6 | 0.285714*a+0.714286*c | 0.446429*(a-c) | 0.127551*(a-c)$^2$ |
| 0.65 | 0.259259*a+0.740741*c | 0.448934*(a-c) | 0.11639*(a-c)$^2$ |
| 0.7 | 0.230769*a+0.769231*c | 0.452489*(a-c) | 0.10442*(a-c)$^2$ |
| 0.75 | 0.2*a+0.8*c | 0.457143*(a-c) | 0.091429*(a-c)$^2$ |
| 0.8 | 0.166667*a+0.833333*c | 0.462963*(a-c) | 0.07716*(a-c)$^2$ |
| 0.85 | 0.130435*a+0.869565*c | 0.470035*(a-c) | 0.061309*(a-c)$^2$ |
| 0.9 | 0.090909*a+0.909091*c | 0.478469*(a-c) | 0.043497*(a-c)$^2$ |
| 0.95 | 0.047619*a+0.952381*c | 0.4884*(a-c) | 0.023257*(a-c)$^2$ |
| 1 | c | 0.5*(a-c) | 0 |

**Appendix C**

$$q^* = \frac{a-c}{(1+d)(2-d)} \to q^{*'} = \frac{\Delta q^*}{\Delta d} = -(a-c)\frac{[(1+d)(2-d)]'}{[(1+d)(2-d)]^2} \quad = -(a-c)\frac{[2-d+(1+d)(-1)]}{(1+d)^2(2-d)^2}$$
$$= \frac{(a-c)(1-2d)}{(1+d)^2(2-d)^2}$$

Excepting the 1-2d term, all other brackets are positive, so the derivate sign is given by its sign. As ½ is the critical value, we get:

$$\begin{cases} 1-2d < 0\ (\forall)d \in [0;\frac{1}{2}) \\ 1-2d > 0\ (\forall)d \in (\frac{1}{2};1] \end{cases} \to \begin{cases} q^{*'} < 0\ (\forall)d \in [0;\frac{1}{2}) \\ q^{*'} > 0\ (\forall)d \in (\frac{1}{2};1] \end{cases} \to \begin{cases} q^* \downarrow (\forall)d \in [0;\frac{1}{2}) \\ q^* \uparrow (\forall)d \in (\frac{1}{2};1] \end{cases}$$

**Appendix D**

$$\pi^* = \frac{(a-c)^2(1-d)}{(2-d)^2(1+d)} \to \pi^{*'} = \frac{\Delta\pi^*}{\Delta d} = (a-c)^2\frac{-(2-d)^2(1+d)-(1-d)[(2-d)^2(1+d)]'}{[(2-d)^2(1+d)]^2} =$$
$$= (a-c)^2\frac{-(4-4d+d^2)(1+d)-(1-d)[-2(2-d)(1+d)+(2-d)^2]}{[(2-d)^2(1+d)]^2}$$
$$= (a-c)^2\frac{-4-4d+4d+4d^2-d^2-d^3-(1-d)(2d^2-2d-4+4-4d+d^2)}{[(2-d)^2(1+d)]^2}$$
$$= (a-c)^2\frac{-d^3+3d^2-4-(1-d)(3d^2-6d)}{[(2-d)^2(1+d)]^2}$$
$$= (a-c)^2\frac{-d^3+3d^2-4-3d^2+6d+3d^3-6d^2}{[(2-d)^2(1+d)]^2} = (a-c)^2\frac{2d^3-6d^2+6d-4}{(2-d)^4(1+d)^2}$$
$$= (a-c)^2\frac{2(d^3-3d^2+3d-2)}{(2-d)^4(1+d)^2} = (a-c)^2\frac{2(d-2)(d^2-d+1)}{(2-d)^4(1+d)^2}$$
$$= -(a-c)^2\frac{2(d^2-d+1)}{(2-d)^3(1+d)^2} < 0 \ \to \pi^{*'} < 0\ (\forall)d \in [0;1) \to \pi^* \downarrow (\forall)d \in [0;1)$$

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